

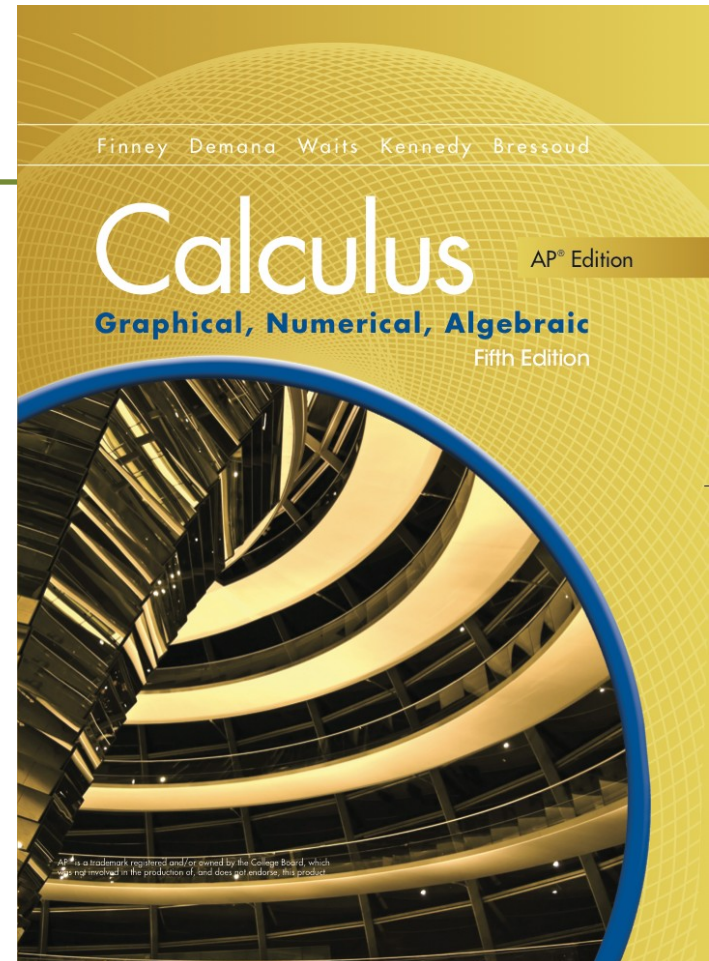
Chapter 8

Applications of Definite Integral



Section 8.3

Volumes



Quick Quiz

1. Simplify the function.

a. $\sqrt{4 + 4x + x^2}$ on $[1, 5]$

b. $\sqrt{1 + \cot^2 x}$ on $[1, 2]$

c. $\sqrt{1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2}$ on $[4, 12]$

2. Identify all values of x for which the function fails to be differentiable.

a. $f(x) = |x - 1|$

b. $f(x) = 3x^{\frac{2}{3}}$

c. $f(x) = \sqrt{1 + 2x + x^2}$

Quick Quiz Solutions

1. Simplify the function.

a. $\sqrt{4 + 4x + x^2}$ on $[1, 5]$ $2 + x$

b. $\sqrt{1 + \cot^2 x}$ on $[1, 2]$ $\csc x$

c. $\sqrt{1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2}$ on $[4, 12]$ $\frac{x^2 + 4}{4x}$

2. Identify all values of x for which the function fails to be differentiable.

a. $f(x) = |x - 1|$ 1

b. $f(x) = 3x^{\frac{2}{3}}$ 0

c. $f(x) = \sqrt{1 + 2x + x^2}$ -1

What you'll learn about

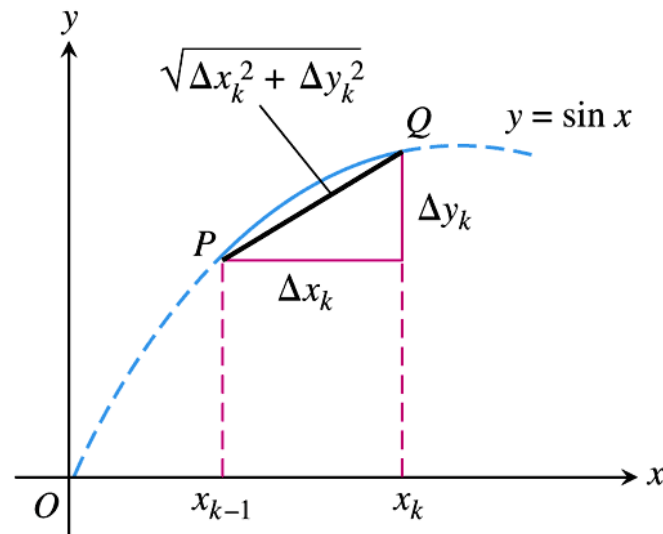
- Lengths of smooth curves
- Lengths of curves with vertical tangents, corners, or cusps

...and why

The length of a smooth curve can be found using a definite integral.

Example The Length of a Sine Wave

What is the length of the curve $y = \sin x$ from $x = 0$ to $x = 2\pi$?



Partition $[0, 2\pi]$ into intervals so short that the pieces of curve lying directly above the intervals are nearly straight. Each arc is nearly the same as the line segment joining its two ends. The length of the segment is $\sqrt{\Delta x_k^2 + \Delta y_k^2}$.

Example The Length of a Sine Wave

The sum $\sum \sqrt{\Delta x_k^2 + \Delta y_k^2}$ over the entire partition approximates the length of the curve. Rewrite $\sum \sqrt{\Delta x_k^2 + \Delta y_k^2}$ as a Riemann

$$\begin{aligned}\text{sum: } \sum \sqrt{\Delta x_k^2 + \Delta y_k^2} &= \sum \frac{\sqrt{\Delta x_k^2 + \Delta y_k^2}}{\Delta x_k} \Delta x_k \\ &= \sum \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k} \right)^2} \Delta x_k\end{aligned}$$

Rewrite the last square root as a function evaluated at some c_k in the k th subinterval. Use the Mean-Value Theorem for differentiable functions to obtain the sum: $\sum \sqrt{1 + (\sin' c_k)^2} \Delta x_k$.

Take the limit as the norms of the subdivisions go to zero:

$$L = \int_0^{2\pi} \sqrt{1 + (\sin' x)^2} dx = \int_0^{2\pi} \sqrt{1 + (\cos x)^2} dx \approx 7.64.$$

Arc Length: Length of a Smooth Curve

If a smooth curve begins at (a, c) and ends at (b, d) , $a < b$, $c < d$, then the length (arc length) of the curve is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \quad \text{if } y \text{ is a smooth function of } x \text{ on } [a, b];$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy} \right)^2} dy \quad \text{if } x \text{ is a smooth function of } y \text{ on } [c, d].$$

Example Applying the Definition

Find the length of the curve $y = x^2$ for $0 \leq x \leq 1$.

$\frac{dy}{dx} = 2x$, this is continuous on $[0, 1]$. Therefore,

$$L = \int_0^1 \sqrt{1 + (2x)^2} \text{ using NINT}$$

$$L \approx 1.479$$

Example A Vertical Tangent

Find the length of the curve $y = x^{\frac{1}{3}}$ between $(-1, -1)$ and $(1, 1)$.

$\frac{dy}{dx} = \frac{1}{3x^{\frac{2}{3}}}$ is not defined at $x = 0$. There is a vertical tangent at $(0, 0)$.

Change to x as a function of y in order to make the tangent at the origin horizontal and the derivative equal to zero instead of undefined.

Solve $y = x^{\frac{1}{3}}$ for x $x = y^3$ and $\frac{dx}{dy} = 3y^2$

$$L = \int_{-1}^1 \sqrt{1 + (3y^2)^2} dy \approx 3.096 \text{ using NINT.}$$

Example Getting Around a Corner

Find the length of the curve $y = |x + 1|$ from $x = -2$ to $x = 1$.

There is a corner at $x = -1$ so neither $\frac{dy}{dx}$ nor $\frac{dx}{dy}$ can exist.

To find the length, split the curve at $x = -1$ to write the function

without absolute values: $|x + 1| = \begin{cases} -x - 1 & \text{if } x < -1, \\ x + 1 & \text{if } x \geq -1. \end{cases}$

Then, $L = \int_{-2}^{-1} (-x - 1) dx + \int_{-1}^1 (x + 1) dx = 2.5$